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S wave kinks of the Dirac–Weyl equation

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Abstract. The S wave kinks of the non-linear equation obtained by Weyl for the Dirac field in a mixed theory of gravitation are calculated numerically. It is found that, for certain values of the parameters, these kinks represent particles similar to baryons, from the kinematical point of view.

1. Introduction

In 1950 Weyl found powerful geometric reasons which suggest that the Dirac equation must include a non-linear term representing a spin–spin self-coupling of gravitational origin. More precisely Weyl studied the Dirac field subject to two conditions: invariance of the theory under the Lorentz gauge group and independence of the metric and affine properties of space–time geometry (Weyl 1950, Kibble 1961, Sciama 1962). The first condition can be satisfied with the introduction in the Dirac Lagrangian of a covariant derivative

$$D_k\psi = h_k{}^{\mu}(\partial_\mu\psi + \tfrac{1}{2}A^{ij}{}_{\mu}S_{ij}\psi),$$

where S_{ij} are the generators of the Lorentz group and $h_k{}^{\mu}$ and $A^{ij}{}_{\mu}$ are geometric fields, which are often called in the physical literature the components of the *vierbein* and local affine connection respectively. They play the same role in the tangent space basis $X_j = h_j{}^{\mu}\partial_\mu$ as $g_{\mu\nu}$, $\Gamma^{\alpha}{}_{\beta\gamma}$ do in the basis $\partial_\mu = \partial/\partial x^{\mu}$. Given $h_k{}^{\mu}$ and $A^{ij}{}_{\mu}$, the curvature and torsion tensors, R and C, are determined using the Cartan structural equations (Helgason 1962).

The second condition implies that $h_k{}^{\mu}$ and $A^{ij}{}_{\mu}$ must be treated as independent quantities in the variational derivation of the field equations. This procedure is called the mixed theory of gravitation (Weyl 1950) and, if only gravitation and electromagnetism are present, is completely equivalent to the more usual metric theory. This is no longer the case if spinor fields are present. After the introduction of the scalar of curvature as the Lagrangian of the geometric fields, Weyl showed that in the flat limit one can go over to the Minkowski space–time if the usual linear Dirac equation is replaced by

$$\mathrm{i}\gamma^{\mu}\partial_\mu\psi - m\psi + 2\lambda(\bar{\psi}\gamma^{\mu}\gamma^5\psi)\gamma_\mu\gamma^5\psi = 0, \tag{1}$$

where $\lambda = -3/16\kappa$, κ being the gravitation constant. Equation (1) will be referred to as the Dirac–Weyl equation. Since the space–time is no longer a Riemannian manifold there is no reason to restrict the geometric Lagrangian to the scalar curvature. An attractive possibility is to include invariants constructed with the torsion tensor

C^i_{jk} (Hayashi 1968). The simplest of these is $C^i_{jk}C^{jk}_i$ which can be shown to lead to the same equation (1) but with a different value of λ (Rañada and Soler 1972). For this reason we will consider equation (1) without any assumption about the value of λ.

First of all, equation (1) can be deduced from the Lagrangian density

$$L = L_{\mathrm{D}} + L_{\mathrm{NL}} \tag{2}$$

where L_{D} is the usual Dirac Lagrangian, and

$$L_{\mathrm{NL}} = \lambda \bar{\psi}\gamma^{\mu}\gamma^{5}\psi\bar{\psi}\gamma_{\mu}\gamma^{5}\psi. \tag{3}$$

Because the four spinors in (3) are equal, L_{NL} can be put in the form (Finkelstein *et al* 1951):

$$L_{\mathrm{NL}} = \lambda \bar{\psi}\gamma^{\mu}\psi\bar{\psi}\gamma_{\mu}\psi. \tag{4}$$

In its two-dimensional version the corresponding theory is called the Thirring model which is at present receiving a great deal of attention (Thirring 1958).

A similar form of equation:

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\psi - m\psi + 2\lambda(\bar{\psi}\psi)\psi = 0 \tag{5}$$

was proposed by Soler (1970) as a model of a three-dimensional elementary particle. In fact, (5) can be obtained by the arguments of Weyl in a simple model of the universe (Rañada and Soler 1972). It has been used to construct some models of the nucleon (Rañada *et al* 1974, Rañada and Vazquez 1976) where the particle appears as a kink, solution of (5) or of an analogous equation which includes a pseudoscalar field to account for the mesonic cloud. The main properties of nucleons are fairly well described, including the sign of the mass difference in one of the models. It would be important to know if these kinks are stable upon collisions, in other words, if they are solitons. This is a very difficult problem although calculations of Soler (1977) indicate a strong degree of stability. In this paper the S wave kinks of the Dirac–Weyl equation will be studied numerically and some of their particle aspects will also be considered.

2. Numerical solution of the Dirac–Weyl equation

The Dirac–Weyl equation is not factorisable in spherical coordinates. For this reason it was solved in the lowest wave approximation, which can be performed by two different methods. The first method consists in making a multipole expansion of the Dirac field and obtaining radial equations for each partial wave. After that, one neglects all the waves except the $S_{1/2}$ one which has the form

$$\psi = \mathrm{e}^{-\mathrm{i}\Omega m t}\left(\frac{m}{2|\lambda|}\right)^{1/2}\begin{pmatrix} G(\rho)\begin{pmatrix}1\\0\end{pmatrix} \\ \mathrm{i}F(\rho)\begin{pmatrix}\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta\end{pmatrix}\end{pmatrix}, \qquad \rho = mr, \tag{6}$$

where G, F are dimensionless fields. The second method is a variational method in which the form (6) of the spinor is substituted in the action integral, the integration over the angles is made and the functions F, G are varied. It has been proved by direct computation that the two methods are equivalent and that they lead to the same equations:

$$
\begin{aligned}
&G' + [1+\Omega+\mathrm{sgn}(\lambda)(F^2-\tfrac{1}{3}G^2)]F = 0,\\
&F' + (2/\rho)F + [1-\Omega+\mathrm{sgn}(\lambda)(\tfrac{1}{3}F^2-G^2)]G = 0.
\end{aligned}
\tag{7}
$$

The boundary conditions are $F(\infty)=G(\infty)=0$, in such a way that

$$\int \psi^+\psi \, \mathrm{d}^3\boldsymbol{r} < \infty.$$

In order to calculate the energy of a solution the energy–momentum tensor was used

$$T^{\alpha\beta} = \tfrac{1}{4}\mathrm{i}[\bar{\psi}\gamma^\alpha\partial^\beta\psi - (\partial^\beta\bar{\psi})\gamma^\alpha\psi - (\partial^\alpha\bar{\psi})\gamma^\beta\psi + \bar{\psi}\gamma^\beta\partial^\alpha\psi] - g^{\alpha\beta}L \tag{8}$$

which gives

$$E = \int T^{00}\,\mathrm{d}^3\boldsymbol{r} = \frac{2\pi}{|\lambda|m}[\Omega I_1 + \mathrm{sgn}(\lambda)(\tfrac{1}{2}I_2 + \tfrac{2}{3}I_3)], \tag{9}$$

where

$$I_1 = \int_0^\infty (F^2+G^2)\rho^2\,\mathrm{d}\rho, \qquad I_2 = \int_0^\infty (F^2-G^2)^2\rho^2\,\mathrm{d}\rho, \qquad I_3 = \int_0^\infty F^2G^2\rho^2\,\mathrm{d}\rho.$$

The norm and spin are given by

$$
\begin{aligned}
&N = \int \bar{\psi}\gamma^0\psi\,\mathrm{d}^3\boldsymbol{r} = \frac{2\pi}{|\lambda|m^2}I_1,\\
&S_k = \tfrac{1}{2}\epsilon_{ijk}\int (x^iT^{j0} - x^jT^{i0})\,\mathrm{d}^3\boldsymbol{r},
\end{aligned}
\tag{10}
$$

which gives

$$S_x = S_y = 0, \qquad S_z = \frac{\pi}{|\lambda|m^2}I_1 = \frac{N}{2}. \tag{11}$$

The existence of solutions of (7) seems to depend on the sign of λ. Let us take the case $\lambda>0$. In order to avoid the singularity at $\rho=0$ equations (7) were regularised by the change $\rho\to\rho^*=\ln\rho$. Two different numerical methods were used, namely a Runge–Kutta fourth-order method and the Hamming predictor–corrector method, and complete agreement between them was found. If the solution is to be regular at the origin, $F(0)=0$. For each value of $G(0)$ a solution is obtained. It turns out that there is a special value of $G(0)$ which separates the solutions with $G(\infty)=+\surd(1-\Omega)$ from those with $G(\infty)=-\surd(1-\Omega)$. This value corresponds to a square integrable solution, in other words, to a finite energy state. This procedure was followed in several related papers (Finkelstein *et al* 1956, Soler 1970), to which we refer for details.

A search for finite energy solutions in the interval $-1<\Omega<+1$, $0<G(0)<10^5$ resulted in a branch of solutions without nodes but failed to find any solution with nodes. If $|\Omega|>1$ the solutions do not decay fast enough at infinity. In figure 1 a typical solution is shown. In figures 2 and 3 we plot $G(0)$ against Ω and $\lambda mE/2\pi$ against Ω. As an indication of the size of the kink the value of $\langle\rho^2\rangle^{1/2}$ is represented in figure 4, as a function of Ω.

The results show some curious features. First of all no solution with nodes was found. Secondly there are several nodeless solutions for the same value of Ω in the

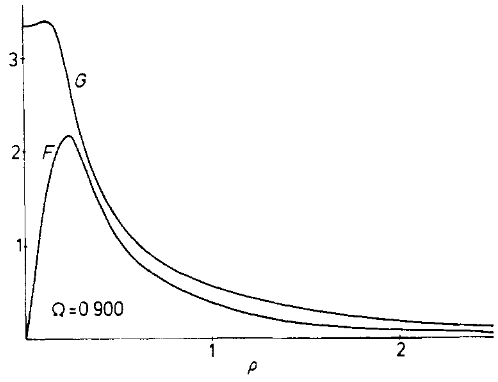

Figure 1. Shape of the radial functions $G(\rho)$, $F(\rho)$ for a typical solution, corresponding to $\Omega = 0{\cdot}9$.

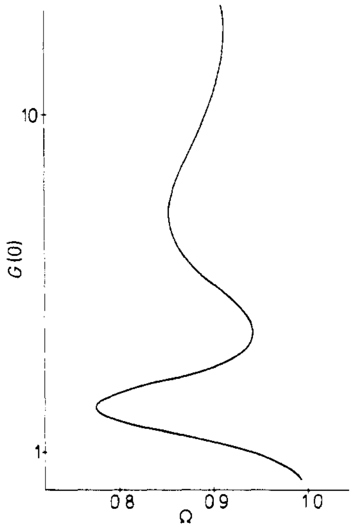

Figure 2. $G(0)$ as a function of Ω.

interval $0{\cdot}78 < \Omega < 0{\cdot}94$. The curve $E(\Omega)$ has a very curious form, a spiral with focus at the point $\Omega = 0{\cdot}88859$, $E = 11{\cdot}6966/\lambda m$. This is in sharp contrast with the $(\bar{\psi}\psi)^2$ Soler model or with the ϕ^4 model. In these cases there seems to exist a branch for each number of nodes and only one solution in the same branch for any prescribed value of Ω, a situation with a certain similarity to that of a typical linear Sturm–Liouville system.

In the particle models mentioned previously the physical values of Ω correspond to the minima of $E(\Omega)$. As we see in this model, there seems to be an infinity of such minima. The lowest three of them correspond to the following values:

$$\Omega = 0{\cdot}899 \qquad E = 9{\cdot}0509/\lambda m$$

$$\Omega = 0{\cdot}892 \qquad E = 10{\cdot}8397/\lambda m$$

$$\Omega = 0{\cdot}893 \qquad E = 11{\cdot}4247/\lambda m.$$

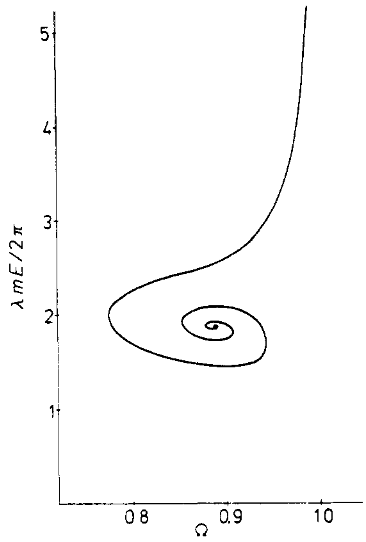

Figure 3. $\lambda mE/2\pi$ as a function of Ω.

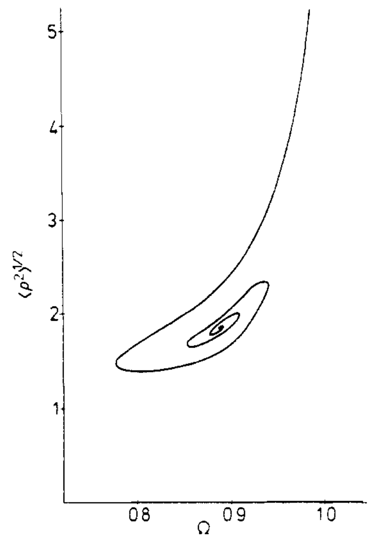

Figure 4. The mean square radius $\langle\rho^2\rangle^{1/2}$ as a function of Ω.

If we try to construct a model of the nucleon we can take $m = 670\,\text{MeV}\,\hbar^{-1}$, $\lambda = 14{\cdot}4\,\text{GeV}^{-2}\,\hbar$. The first minimum then corresponds to:

$$E = M_{\text{p}}; \quad N = \hbar; \quad \langle r^2\rangle^{1/2} = 0{\cdot}5 \text{ fermi}; \quad S_z = \hbar/2,$$

where M_{p} is the proton mass. The second and third minima have energies of 1123 MeV and 1184 MeV which fall very close to the masses of the Λ (1115 MeV) and Σ^0 (1192 MeV). It is perhaps a numerical coincidence, but it is interesting to find a spectrum of states with masses of the right order of magnitude. We must stress that the different minima cannot be considered as the usual excited states since they have no nodes. In this aspect the Soler model again has a different behaviour, as the mass of the second minimum (with one node) is about eight times bigger than that of the ground state.

In the case of $\lambda < 0$ no solutions were found in the interval $0 < G(0) < 2 \times 10^5$. This suggests that they do not exist.

3. Conclusion

In conclusion, the independent consideration of the metric and affine properties of space–time together with the assumption of Poincaré gauge invariance of the theory, proposed by Weyl, implies the existence of spinor kinks, perhaps even solitons, which for some values of the parameters have some similarities with the baryons.

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